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THE CONCEPT OF ACCUMULATION IN CALCULUS[†]

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The concept of accumulation is central to the idea of integration, and therefore is at the core of understanding many ideas and applications in calculus. On one hand, the idea of accumulation is trivial. You accumulate a quantity by getting more of it. We accumulate injuries as we exercise. We accumulate junk as we grow older. We accumulate wealth by gaining more of it. There are some details to consider, such as whether it makes sense to think of accumulating a negative amount of a quantity, but the main idea is straightforward.

On the other hand, the idea of accumulation is anything but straightforward. First, students find it is hard to think of something accumulating when they cannot conceptualize the "bits" that accumulate. To understand the idea of accomplished work, for example, as accruing incrementally means that one must think of each momentary total amount of work as the sum of past increments, and of every additional incremental bit of work as being composed of a force applied through a distance. Second, the mathematical idea of an accumulation function, represented as $F(x)=\int_a^x f(t)\,dt$, involves so many moving parts that it is understandable that students have difficulty understanding and employing it.

Readers already sophisticated in reasoning about accumulations may find it surprising that many students are challenged to think mathematically about them. The ways in which it is difficult, though, are instructive for a larger set of issues in calculus. As such, our intention in this chapter is to:

(1) Explicate the complex composition of a well-structured understanding of accumulation functions,
(2) Illustrate students' difficulties in understanding accumulation mathematically,
(3) Point out promising approaches in helping students conceptualize accumulation functions, and
(4) Place students' understandings of accumulation functions within the calculus as a whole.

Composition Of A Well-Structured Understanding Of Accumulation Functions

Accumulation functions can be represented generally by $\int_a^x f(t)\,dt$.[1] It is worthwhile to unpack the meanings behind this formula in order to see all that it entails. We will do this in two passes, first without addressing ideas of Riemann sums, then addressing them.

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[1] For purposes of this chapter, we will speak only of Riemann integrals over an interval.

Accumulation Functions

For sake of illustration, let $f(x) = 2e^{-\cos(x)} - 2.5$. To understand an accumulation function involving *f*, students must have a *process conception* of the formula $2e^{-\cos(x)} - 2.5$.[2] This means that they must hold the perspective that though it might require actual effort to calculate any particular value of this formula, in the end it represents a number, and the number it represents depends only on the value of *x* (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; see also Oehrtman, Carlson, & Thompson, this volume). To have a process conception of a function's defining formula implies that one has what Gray and Tall (1994) call a *proceptual* understanding of what the formula represents. One has in mind a well-structured set of procedures for evaluating the formula together with ability and inclination to see the formula as "self-evaluating" (P. W. Thompson, 1994b), meaning that one sees it as evaluating itself instantaneously for any number.[3]

To understand an accumulation function, students also need a covariational understanding of the relationship between *x* and *f* (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; P. W. Thompson, 1994b, 1994c). In the case of the current example, this means understanding that as the value of *x* varies, the value of $2e^{-\cos(x)} - 2.5$ varies accordingly. It also entails creating an image of *how* the value of $2e^{-\cos(x)} - 2.5$ varies as the value of *x* varies, thus generating the relationship expressed by the graph in Figure 1.

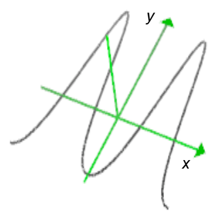

Figure 1: Graph that depicts values of x and $2e^{-\cos(x)} - 2.5$ varying simultaneously.

Students who have mastered the process conceptions of formulae and covariational conceptions of function must then coordinate a third aspect with them—imagining accumulation and its quantification. Students must coordinate the value of *x* as it varies from some starting point, the value of $2e^{-\cos(x)} - 2.5$ as it varies accordingly, and, in addition, imagine the bounded

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[2] Briedenbach et al., Dubinsky and Harel, and Carlson speak of a *process conception of function.* We also speak of a *process conception of formulae.* To us, for students to have a process conception of "*f*(*x*) = …" requires that they have a process conception of the right hand side. A process conception of a function entails more than does a process conception of a formula. Our intent is to avoid adopting an "all or none" stance toward what it means to understand a function.
[3] We will give specific examples later in this article of students having and not having a process conception of an integral.

area accumulating (Figure 2) as x and $2e^{-\cos(x)} - 2.5$ vary. Moreover, the student must attend to *how* these values are varying in tandem.[4]

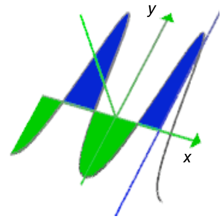

Figure 2. Area accumulates as x varies.

To conceive of an *accumulation function* defined in x is to imagine a total accumulated area for each value of x. This introduces a third dimension into the conceptualization of accumulation functions—students must coordinate three values simultaneously: x, $2e^{-\cos(x)} - 2.5$, and $\int_a^x \left(2e^{-\cos(t)} - 2.5\right) dt$. Figure 3 illustrates this coordination graphically. Points on the space curve are ordered triplets (x, $2e^{-\cos(x)} - 2.5$, $\int_a^x \left(2e^{-\cos(t)} - 2.5\right) dt$). While we do not claim that students must conceptualize the space curve in Figure 3 in order to understand the mathematical idea of accumulation, we do claim that expecting them to understand accumulations *as functions* is tantamount to expecting them to understand a space curve.[5]

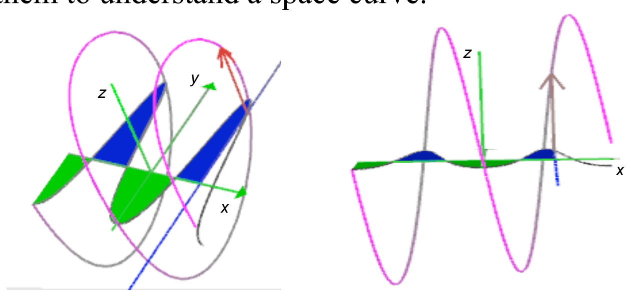

Figure 3: Coordination *of* x, $2e^{-\cos(x)} - 2.5$, *and* $\int_a^x \left(2e^{-\cos(t)} - 2.5\right) dt$, and its projection into the x-z plane.

Before covering these issues again from the perspective of Riemann sums, we would like to point out a notational issue. The role of t in the expression $\int_a^x f(t)\,dt$ often is a mystery to students. When textbooks address it at all they treat it as a "dummy variable," or a variable that will disappear when the expression is evaluated (Weisstein, 2006). We propose that t actually

[4] A simpler example involves coordinating changes in x with changes in $3x^2$ while simultaneously imaging how the bounded area under the graph of $3x^2$ is accumulating.
[5] A colleague thought we are implying that accumulation functions should therefore not be taught, since they are so sophisticated. On the contrary, we argue that they *should* be taught, but they should be taught with full awareness of what it means to understand them.

serves a conceptual role in making sense of the expression $\int_a^x f(t)\,dt$. In **Figure 2**, *f* cannot be thought of as having the same argument as does $\int_a^x f(t)\,dt$. In a sense, the graph of *f* must "pre-exist" when imagining the accumulation of area between it and an axis. Thinking of *t* as already having varied through *f*'s domain prior to *x* varying through a subset of *f*'s domain then allows one to think of $\int_a^x f(t)\,dt$ as representing the accumulation of area within an already bounded region.

Riemann Sums

Calculus texts typically offer Riemann sums as a way to approximate areas bounded by a curve. The question of how a bounded area itself can represent a quantity other than area requires us to examine ways to understand Riemann sums and how they arise.

If *f* is a function whose values provide measures of a quantity, and *x* also is a measure of a quantity, then $f(c)\Delta x$, where $c \in [x, x+\Delta x]$, is a measure of a derived quantity. The simplest case is when $f(x)$ is a measure of length and *x* is a measure of length. Then $f(c)\Delta x$ is a measure of area. If $f(x)$ is a measure of speed and *x* is a measure of time, then $f(c)\Delta x$ is a measure of distance. If $f(x)$ is a measure of force and *x* is a measure of distance, then $f(c)\Delta x$ is a measure of work. If $f(x)$ is a measure of cross-sectional area and *x* is a measure of height, then $f(c)\Delta$x is a measure of volume. If $f(x)$is a measure of electric current and *x* is a measure of time, then $f(c)\Delta x$ is a measure of electric charge. A Riemann sum, then, made by a sum of incremental bits each of which is made multiplicatively of two quantities, represents a total amount of the derived quantity whose bits are defined by $f(c)\Delta x$. Therefore, for students to see "area under a curve" as representing a quantity other than area, it is imperative that they conceive of the quantities being accumulated as being created by accruing incremental bits that are formed multiplicatively.

Our account of how "area under a curve" comes to represent quantities other than area clearly holds an undertone of thinking with infinitesimals. Though a large portion of 19th-century activity in the foundations of mathematics was motivated by the desire to eliminate infinitesimals, we see no way around explicitly supporting students' reasoning about them as part of their path to understanding accumulation functions (and functions in general). Much of this support should be given in middle school and high school, but given that this does not happen in the United States, it must be addressed in introductory calculus courses.[6]

Students' Difficulties in Understanding Accumulation Mathematically

While our analysis of the ideas entailed in understanding accumulation functions also points to ways that students have difficulty with them, we must also note that the major source of

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[6] The legacy of infinitesimal reasoning in calculus is reflected in the continued use of the integral notation that Leibniz introduced in 1675, when he used "*dt*" in $\int f(t)\,dt$ to represent the difference between successive values of *t* (O'Connor & Robertson, 2005).

students' problems with the idea of accumulation functions is that it is rarely taught with the intention that students actually understand it. We anticipate the objection that definite integrals already receive clear and explicit attention in every calculus textbook. Our reply is that a definite integral is to an accumulation function as 4 is to x^2. No one would claim to teach the idea of function by having students calculate specific values of one. Similarly, we should not think that we are teaching the idea of accumulation function by having students calculate specific definite integrals.

We say this without hubris. In a teaching experiment (P. W. Thompson, 1994a) conducted with the intention of investigating advanced undergraduates' difficulties forming the ideas of accumulation and rate of change of accumulation, the first author failed to anticipate the difficulty they would have conceptualizing accumulation functions and failed to anticipate the importance that they actually do so for understanding the Fundamental Theorem of Calculus (FTC).[7]

Lastly, the idea of limit and the use of notation are two of the most subtle and complex aspects of understanding accumulation functions. Research on students' understanding of limit (Cornu, 1991; Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Tall, 1992; Tall & Vinner, 1981; Williams, 1991) shows consistently that high school and undergraduate students understand limits poorly, even after explicit instruction on them. We located only two empirical studies that addressed students' reasoning about limit in the context of integration (Oehrtman, 2002; P. W. Thompson, 1994a). Thompson studied advanced undergraduate and graduate students' understanding of the FTC, and in that context found students concluding, for example, that the rate of change of volume with respect to height in a cone was equal to the cross-sectional area at that height because *as you make an increment in height smaller, the incremental cylinder of volume gets closer and closer to an area* (P. W. Thompson, 1994a, p. 34). Oehrtman (2002) named this way of thinking "the collapsing metaphor," meaning that students reasoned that the object being considered (e.g., a cone, a secant, etc.) approached another object having one less dimension. He found one-third of his subjects (first-year calculus students after instruction) employing this metaphor in one setting or another.

Oehrtman points out that though the collapsing metaphor is mathematically incorrect, it sometimes enables students to educe mathematically correct results from incorrect reasoning. Students sometimes justify the FTC by the incorrect reasoning that as the interval width decreases, the rectangle collapses to its height (Figure 4). Put another way, students reason that $\Delta x \to 0$ implies that $f(c)\Delta x \to f(c)$. They were thinking of an image (e.g., a rectangle) instead of the quantity (e.g., electrical charge) and the value of its measure. We do note that although the collapsing metaphor enables students' intuitive, albeit incorrect, "justification" of the FTC, it also divorces their understanding of the fundamental theorem from any idea of rate of change.

[7] This was a classic case of an outcome being harder than someone expected even though he anticipated it would be harder than he expected.

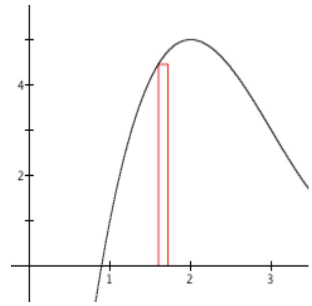

Figure 4. As Δx approaches 0, f(c)Δx approaches f(c).

Finally, we address the issue of what one might take as evidence as to whether students understand the representation of an accumulation function. The goal is to avoid accepting as evidence of understanding what Vinner (1997) calls pseudo-analytic and pseudo-conceptual behavior. Students exhibit conceptual behavior when their words and symbols refer to ideas and relationships. They exhibit pseudo-conceptual behavior when their words and notations refer to other words, to notations, or to iconic images. They exhibit pseudo-analytic behavior by applying pseudo-conceptual thinking in the course of their reasoning. The following student's response to the prompt, "Explain what $\int_a^x f(t)dt$ means," illustrates the subtleness in distinguishing between conceptual and pseudo conceptual behavior.

> $\int_a^b f(x)dx$ gives the area bounded by the graph of $f(x)$ and the lines $y=0$, $x=a$, and $x=b$. Therefore, $\int_a^x f(t)dt$ gives the area bounded by $f(t)$, $y=0$, $t=a$, and $t=x$. As x varies, the bounded area varies.

This answer, on the surface, appears quite acceptable. The problem is that we cannot tell which of several possible meanings this student gives to the integral notation. We highlight this by making a notational substitution in this student's answer so that it responds to the question, "Explain what $A_a^x\left(f(t)\right)$ means when $A_a^b\left(f(x)\right)$ stands for the area bounded by the graph of $f(x)$ and the lines $y=0$, $x=a$, and $x=b$."

> $A_a^b\left(f(x)\right)$ gives the area bounded by the graph of $f(x)$, $y=0$, $x=a$, and $x=b$. Therefore, $A_a^x\left(f(t)\right)$ gives the area bounded by $f(t)$, $y=0$, $t=a$, and $t=x$. As x varies, the bounded area varies.

In other words, students could be imagining no more than a concrete image of a region "filling up" with paint as one moves one of its vertical edges (Figure 5), and at the same time could be using integral notation referentially (Figure 6) to describe that image. While this understanding of $\int_a^x f(t)dt$ would be a process conception of the notation, and indeed is useful as a shorthand for anyone having an in-depth understanding of the function, the process students conceive when they have *only* the shorthand has nothing to do with the meaning of integration as the limit of Riemann sums.

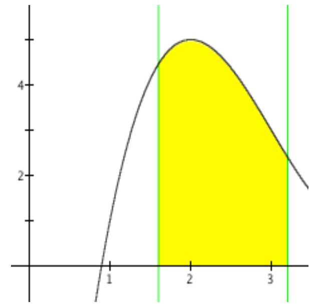

Figure 5. A student's image of a region filling up with paint as she moves one edge.

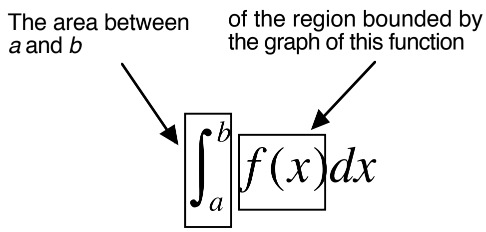

Figure 6. Referential understanding of integral notation.

We mentioned earlier a teaching experiment that investigated difficulties inherent in coming to understand the FTC (P. W. Thompson, 1994a). It involved 19 advanced undergraduate mathematics and masters mathematics education students. One aspect of the teaching experiment emphasized students' development of a process conception of integrals that entailed ideas of accumulation, variation, and Riemann sums as the root ideas of integration. We joined these ideas by defining Riemann sums as one would for fixed intervals, but modifying the definition so that Δx was a parameter and x was a variable. That is, we held Δx constant and let x vary instead of holding x constant and letting Δx vary.[8] This can be expressed generally as $F_{\Delta x,a}(x) = \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} f(i\Delta x + a)\Delta x, a \le x \le b$. However, we did not provide this general representation of the Riemann accumulation function. Instead, we worked with students to model the accumulation of a quantity that accrued in bits created by joining two of its constituent attributes (like work from power and time) and develop a representation of the total accumulation.

At the end of the teaching experiment we used Item 6, among others, to assess the extent to which students had developed a process conception of Riemann accumulation functions.

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[8] Our justification for this approach, fixing Δx and letting x vary, was our intention to have students understand integral accumulation functions as being rooted in Riemann accumulation functions. In earlier attempts to probe students' understandings of accumulation functions, we got only the "paint filling" metaphor alluded to in the discussion of **Figure 5**. We will say more about the benefits of this approach in the last section.

Item 6

Let $q(x)$ be defined by $q(t)=\sum_{i=1}^{t/\Delta t} f(i\Delta t)\Delta t$, $0 \le t \le b$. Explain the process by which the expression $\sum_{i=1}^{t/\Delta t} f(i\Delta t)\Delta t$ assigns a value to $q(t)$ for each value of t in the domain of f.

Responses to this item showed that understanding Riemann sums as functions was a complex act for students. Student 1 wrote:

> First the value of a certain chunk is measured by $i\Delta t$. This is then multiplied by the change which is Δt. This is repeated for every value of t and then added up. Each value of t is cut up into $t/\Delta t$ intervals, and added. $t/\Delta t$ is the number of intervals the piece is to be divided up into.

Student 1 had a number of problems, one being that he was imagining a "chunk" of a quantity independently of an analytic expression that established its measure—$i\Delta t$ does not "measure" the chunk, it just puts you at the right place to make it. The student also failed to note the role played by the function f in creating a "chunk"— it is $f(i\Delta t)\Delta t$ that gives the chunk's measure. Also, the student was unclear about what was being summed: "Each value of t is cut up into $t/\Delta t$ intervals, and then added." However, the subintervals are not summed. A more serious problem, though, is that this student appeared to be imagining t and i varying simultaneously instead of, first, varying i from 1 to $\lfloor t/\Delta t \rfloor$ for a fixed value of t, and then varying t.

Student 2 wrote:

- Here Δt represents the size of each interval that f is being broken up into.

- So $t/\Delta t$ equals the number of intervals the graph of f is broken up into.

- So our i starts out at 1 and then goes to $t/\Delta t$.

- The expression first finds f and then it finds the ith interval of f that we are dealing with. Then it finds the value of the function f at that interval and then multiplies by Δt. This finds the area of that particular rectangle. Then we add it to the previous areas found and plot that point. You then connect all the points to get your curve.

The process that Student 2 understood is much more well-structured than Student 1's. While some of her phrasing is imprecise ("… the size of each interval that *f is being broken up into*") and suggests that she is reasoning about a graph, she does seem to be imagining the process being played out for each value of t. One missing element in this student's explanation is that the value of t does not vary. Rather, she seems to imagine that she "samples" values of t and then connects the points that get plotted for each value. This suggests that she was imagining a Riemann sum over a fixed interval, which would normally correspond to an approximation of a definite integral $\int_a^b f(t)dt$ instead of the indefinite integral $\int_a^x f(t)dt$. When students do not see the upper limit as varying, it is difficult, if not impossible, for them to conceive that the accumulation function has a rate of change for every value at which it is defined (Smith, in preparation).

Only seven of the 19 students expressed an appropriate order of variation for the index variable of the Riemann sum and the argument of the function. Five students appeared to have mixed images of definite and indefinite integrals (e.g., *i* varied, but *t* did not). The remaining seven students had confounded the two variations so that everything was happening at once. For us, these results imply that the idea of accumulation function is far more complex than is commonly assumed and that it is still unclear what instructional trajectories will best support students learning them.

We note in closing that Item 6, above, was useful in seeing the extent to which students had developed a process conception of Riemann sums and Riemann accumulation functions. In subsequent years, we have found that Item 6′, below, is a better task for determining that students have developed a process conception of a Riemann accumulation function.

> Item 6′.
> Suppose *f* is continuous real-valued function on (a,b) and $\Delta x > 0$.
> Let *g* be defined as $g(x) = \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} f(i\Delta x + a)\Delta x, a \le x \le b$. Explain why *g* is a step function.

To see why *g* is a step function, let x_0 be $2\Delta x + a$. Then $\left\lfloor \frac{x-a}{\Delta x} \right\rfloor$ is 2 for every value of x in $[x_0, x_0 + \Delta x)$, and thus $g(x)$ is $f(0\Delta x + a)\Delta x + f(1\Delta x + a)\Delta x + f(2\Delta x + a)\Delta x$ for every value of x in $[x_0, x_0 + \Delta x)$. Therefore, $g(x)$ is constant as x varies within each interval $[i\Delta x + a, (I + 1)\Delta x + a)$.

Promising Approaches In Helping Students Conceptualize Accumulation Functions

Carlson, Persson, and Smith (2003), building upon Thompson's (1994a) work, conducted a teaching experiment with first-semester calculus students using instruction that addressed the conceptual difficulties experienced by Thompson's students in learning accumulation functions and their rate of change. Their approach to teaching accumulation functions was embedded in a larger effort to have students conceptualize the FTC as the course's culmination. The course began with a review of functions that emphasized covariation of quantities and then leveraged that reasoning to develop rates of change, limits, derivatives, and accumulations in terms of covarying quantities. Carlson et al. spent six sessions over two weeks developing notions of accumulating quantities and accumulation functions and another five sessions over 10 days on the FTC. Their coverage of the FTC had students examine in detail the incremental accumulation of various quantities, tying the idea of accumulation to the notation by which it is represented. They then had students examine the rates at which total accumulations changed by looking at average rates of change of the total accumulation over the interval of incremental accrual.

Carlson et al. reported that their approach led to a high success rate, both in terms of students' conceptions of accumulation functions and their ability to use and explain the FTC.[9]

At the same time that Carlson et al.'s (2003) findings point to the promise of building students' understanding and skill with calculus on a strong conceptual foundation of covariation, function, rate of change, and accumulation, their post-instruction interviews suggest that students had not clarified some important issues. For example, one student, Chad, was shown a graph of a piecewise-linear function *f* whose values gave the rate in gallons per hour at which water filled a container. He was asked to explain the meaning of $g(x)=\int_0^x f(t)\,dt$ and how to evaluate *g*(9). Carlson et al. reported that Chad gave an acceptable explanation of the meaning of *g*(*x*) and also provided Chad's explanation of how to evaluate *g*(9).

1. Interviewer: So, how do you think about evaluating *g*(9)?

2. Chad: I see that as finding the time that passes from 0 to 9 and thinking about how much area gets added under the curve as I move along. I see that water is coming into the tank, first at an increasing rate, then at a decreasing rate. Then after 4½ hours, water starts to go out of the tank. As you add up the area under the curve you see that the same amount of water comes in between 0 and 4 ½ that goes out between time 4½ and 9 …. so, the result is that there is no water in the tank after 9 hours have passed.

3. Interviewer: How are *g* and *f* related?

4. Chad: The derivative of *g* gives the graph of *f*. What I don't get is why *t* is the variable that is used in *f*. I never really understood this on some of the other problems we did either. (Carlson et al., 2003, p. 270)

Paragraph 2 suggests that Chad could think about accumulation functions and rate of change to support his evaluation of *g*(9). He also was thinking about the quantities that *x*, *f*(*x*), and *g*(*x*) represented (*viz.*, number of hours, the rate at which water filled the container, and the amount of water in the container). He also appeared to attend to how *g* changes while imagining changes in *x* and *f*. However, we observe that Chad's statements in paragraph 2 also are reminiscent of the "paint filling" notion of accumulation discussed in conjunction with Figure 5. As a result, without querying Chad further, we have no way of knowing if he is using the "paint metaphor" in a pseudo conceptual way—i.e., does he understand that infinitesimal amounts of multiplicative bits are being accrued as *x* varies. In addition, paragraph 4 suggests that Chad had not worked through the conceptual issues behind the use of *t* in the accumulation function's definition.

It is unclear to us the extent to which Chad's understanding was rooted in Riemann sums as opposed to being rooted in the paint-filling metaphor. We see this as once again pointing to the need for further analysis of what it means for students to understand accumulation functions and how to assess their levels of understanding. It also points to the need for further investigation into the implications that various ways of understanding accumulation have for learning related ideas in the calculus, and the kinds of instruction that will support students in developing those understandings.

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[9] We do not compare performances of Thompson's and Carlson's students because the two studies used different assessment tasks and therefore the results are not directly comparable.

Students' Understandings of Accumulation Functions Within The Calculus as a Whole

Accumulation functions would not be important if understanding them well did not pay off elsewhere. In this section we argue that the kind of understanding we have depicted as well-structured not only pays off in other areas, they are part of understanding many related ideas and they are essential for understanding many advanced ideas in the calculus. But even beyond the connections with other ideas that we will outline here, we feel that the precise thinking and thoughtful use of notation required to understand accumulation functions well is in itself valuable mathematical activity.

Connections With Other Ideas

Rate of change. The idea of accumulation both grows out of and contributes to a coherent understanding of rate of change (Carlson et al., 2003; P. W. Thompson, 1994a). When something changes, something accumulates. When something accumulates, it accumulates at some rate. To understand rate of change well, then, means that one sees accumulation and its rate of change as two sides of a coin. Thus, students' success in the integral calculus can begin in middle school if rate of change is taught substantively (A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994a, 1994c; P. W. Thompson & Thompson, 1994).

Function. The obvious connection between the ideas of accumulation functions and function is that an accumulation function is precisely that, a function. It is nontrivial for students to understand this. There are three additional important connections that can be exploited in a calculus curriculum.

- Accumulation functions are, in all likelihood, the first functions students meet that are defined in terms of a complex process instead of in terms of an algebraic, trigonometric, or exponential expression. The challenge is to avoid leading them to pseudo-conceptual understandings of accumulation (see discussion of Figure 5) and pseudo-analytic interpretations of the notation (see **Figure 6**).
- Accumulation is the root of accumulation functions, and hence covariational conceptions of functions must be a key connection.
- Riemann accumulation functions that are specified by the formula $g(x)=\sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} f(i\Delta x+a)\Delta x, a \le x \le b$ are step functions. Computer programs that allow Riemann sums as defined here will also support students' explorations of convergence.[10] The issue of convergence, however, expresses itself differently in this context than it does in typical treatments of Riemann sums. In the typical case, the issue is whether there is a *number* that is the limit of a Riemann sum as $\Delta x \to 0$. The accumulation function $\int_a^x f(t)\,dt$ is then defined so that each value of the function is a pointwise limit. In the case of a Riemann accumulation function, the issue is whether there is a *function* that is the limit of the family of Riemann accumulation functions that is generated as Δx

[10] We used Graphing Calculator from PacificTech to generate the graphs of accumulation functions contained in Figures 2-5. The functions graphed in those figures were specified as Riemann accumulation functions.

approaches 0. That is to say, the idea of function is always at the forefront in this approach, even when working with approximations. We anticipate the objection that the issue of limits of function sequences is beyond first-term calculus students' conceptual capacity. Our experience is quite the contrary. Students find it visually compelling when they see a sequence of graphs approaching what appears to be the graph of *some* function, and willingly entertain the question, "What is the function that this sequence appears to approach? Does it approach it pointwise or uniformly?[11] How can we determine it analytically?" For example, the graphs of cos(x) and

$g(x)=\sum_{i=0}^{\left\lfloor \frac{x-a}{0.01} \right\rfloor}\cos(0.01i+{}^{-}7)(0.01),\ {}^{-}7\le x<\infty$ are given in **Figure 7**. The accumulation function's graph appears to be of a trigonometric function, but which one? What are its coefficients? Is something added? How is it related to cos(x)? We feel that these questions can be mined fruitfully to develop students' understandings of a web of related ideas—approximation, limits, functions, convergence, and antiderivate, to name a few. How these connections might be developed instructionally, though, requires further investigation.

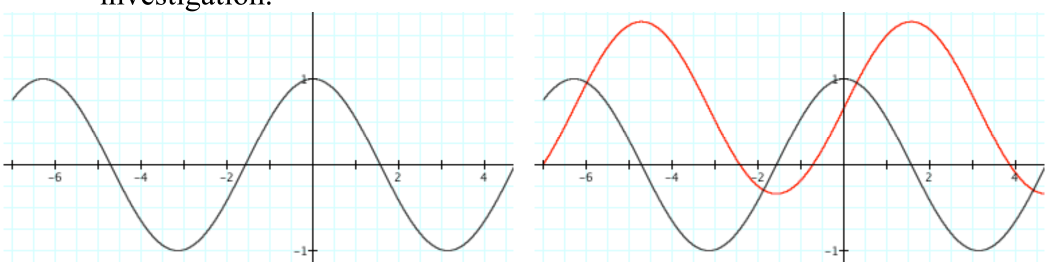

Figure 7. Graphs of cos(x) and its Riemann accumulation function.

Functions of two variables. Carlson, Oehrtman, and Thompson (this volume) argue that ideas of function-as-covariation in the case of two variables can be extended naturally to functions as covariation in the case of three variables. Were one to take that approach, then accumulation functions defined over lines and regions in a plane and over surfaces would be natural extensions from accumulation functions defined over intervals. We suspect that one needn't fall back to Riemann accumulation functions to make these cases meaningful if students have understood them thoroughly when they were first taught. We stress that ideas of covariation must remain at the forefront even with multiple integrals, and the importance of attending to issues of scope of variation is even greater than what we saw earlier in students' understanding of Riemann accumulation functions over intervals.

[11] The distinction between pointwise and uniform convergence arises quite naturally in classroom discussions when looking at the behavior of Riemann accumulation functions for functions with unbounded derivates.

Conclusion

As we mentioned in the beginning of this chapter, the concept of accumulation is almost trivial yet, at the same time, quite complex. One aspect of the complexity described in this chapter was the focus on accumulation functions as opposed to the traditional focus on the calculation of a number representing the area bound by the curve over a specific interval. The emphasis of this chapter, though, was not on the differences between these two related notions of integral calculus; it was on the underlying images students bring to bear on such problems and the implications of those images. The first image involved covering a region, where the result was a number equivalent to "the amount of paint needed" to cover the area between the x-axis and the function on the interval [a,b]. The second image involved measuring the accumulation of a quantity that is created from bits that themselves are made from measures of two quantities, one whose measure is a function of the other on the interval [a,b], by summing values of $f(c)\Delta x$, $c \in [i\Delta x,(i+1)\Delta x)$. The connection between the second image and area is simply that if $f(c)$ and Δx are represented by lengths, then $f(c)\Delta x$ gives the area of a rectangle made from those lengths. The former image is difficult to apply to quantities other than area, while the second necessitates understandings that both $f(c)$ and Δx can be measures of quantities (for example force and distance) and $f(c)\Delta x$ is a measure of a derived quantity (work).

It could appear that these images (painted area and accumulated quantities) are the same. We note in reply that they are only the same when one has constructed a scheme of understandings within which area can represent a quantity other than area. Further, the intricacies of understanding accumulation are often reduced to calculating products and limits without understanding the significance of either. Without additional focus on constructing, representing, and understanding Riemann Sums, there is little reason to believe that students will understand accumulation functions as playing a central role in the FTC.

This paper presents a call for increased emphasis on the FTC as explicating an inherent relationship between accumulation of quantities in bits and the rates at which an incremental bit accumulates. Understanding this relationship entails a clear emphasis on covariation as a foundational idea in calculus instruction. We make this call with awareness of the difficulties involved in developing a well-structured understanding of accumulation functions, and that this difficulty stands in contrast with the efficiency of teaching students to calculate definite integrals as area under a curve. We believe that the benefits make the effort worthwhile. Understanding $\int_a^b f(x)dx$ as an expression that yields the area bound by the x-axis and $f(x)$ is efficient but not generative. It supports a superficial understanding of $\int_a^x f(t)dt$. We believe that understanding accumulation so that $\int_a^b f(x)dx$ is simply $\int_a^x f(t)dt$ evaluated at $x=a$, where $\int_a^x f(t)dt$ itself has a well-developed meaning, can be part of a coherent calculus that focuses on having students see connections among rates of change of quantities, accumulation of quantities, functions as models, limits, antiderivatives, pointwise and uniform convergence, and functions of two (or more) variables. Though more work is needed to flesh out instruction that achieves this, we believe that a focus on accumulation functions as discussed in this chapter will be central to it.

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