

# Worksheet 3

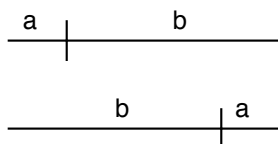
## Properties of Operations

When something is always true about an operation it is called a *property* of that operation. Addition and multiplication have several properties. One is called *commutativity*; the other is called *associativity*.

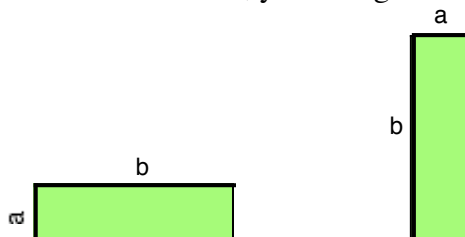
### Part I: Commutative Property of Addition Commutative Property of Multiplication

When we write  $(5+4)=(4+5)$ , we are stating that when we calculate the two expressions we will get the same number. When we write  $(x+y)=(y+x)$  we are stating that regardless of the numbers we add, if we change their order, when we add them we will get the same number.

The commutative property of addition says that you may add quantities in any order and still get the same result. As below, when you have two lengths  $a$  and  $b$ , you get the same length whether you put  $a$  first or  $b$  first. The result,  $(a + b)$  is the same either way.



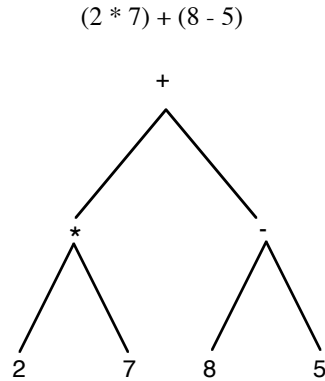
The commutative property of multiplication says that you may multiply quantities in an order and you will get the same result. This is illustrated by computing the area of a rectangle. It doesn't matter which side you consider the width, you will get the same area either way.



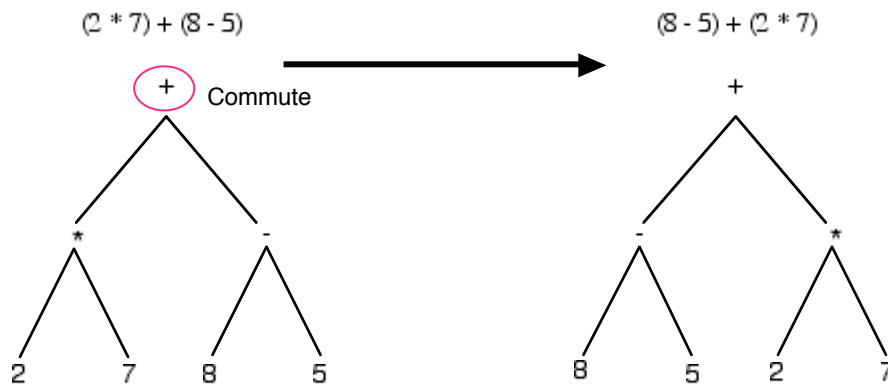
You can change the order in which you add or multiply two numbers and the expression's value will not change. That is, you can change  $\diamond + \Delta$  to  $\Delta + \diamond$ , or you can change  $\diamond * \Delta$  to  $\Delta * \diamond$ , and the result will not change.

Please keep in mind that  $\diamond$  and  $\Delta$  do not have to be numbers. They can be expressions that evaluate to a number. The next example shows this.

Start with  $(2 * 7) + (8 - 5)$ . The tree diagram for this expression shows that it is the sum of  $2*7$  and  $8-5$ .

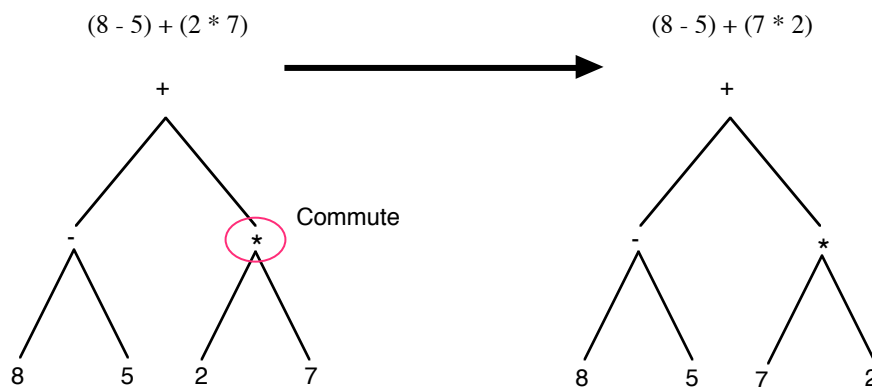


We can use the commutative property of addition to change  $(2 * 7) + (8 - 5)$ , into  $(8 - 5) + (2 * 7)$ .



The result is the expression  $(8 - 5) + (2 * 7)$ . This means that the expression  $(8 - 5) + (7 * 2)$  is *equivalent* to  $(8 - 5) + (7 * 2)$ , because they still evaluate to the same number, namely 17.

You can also apply the commutative property of multiplication to the sub-expression  $(2 * 7)$ .



The result is the expression  $(8 - 5) + (7 * 2)$ . This means that the expression  $(8 - 5) + (7 * 2)$  is *equivalent* to  $(8 - 5) + (2 * 7)$ , because they still evaluate to the same number, namely 17.

Not every operation is commutative. Subtraction is not commutative.  $8 - 5$  is not the same number as  $5 - 8$ . The expression  $8 - 5$  is 3. The expression  $5 - 8$  is -3. That is, *subtraction is not commutative, because changing the order of the operands may change the value of the expression.* What other operations are not commutative?

### Your Turn

Apply COMMUTE to each of the following expressions. Circle the operation to which you will apply the COMMUTE operation, then write the expression and draw the expression tree as they will appear afterward.

a.  $(8 - 5) - (2 * 7)$  → \_\_\_\_\_

b.  $(u + 5) / (2 / 7)$  → \_\_\_\_\_

c. On a separate sheet: Apply COMMUTE to the expressions below that allow it (and write the resulting expression). Explain why the other expressions do not allow COMMUTE.  
NOTE: It might be useful to draw expression trees before deciding.

i)  $a / b - 4$

ii)  $(x / y) + (u \wedge v)$

iii)  $3 + k / (m * 5)$

iv)  $5 * y + u / t + 7$

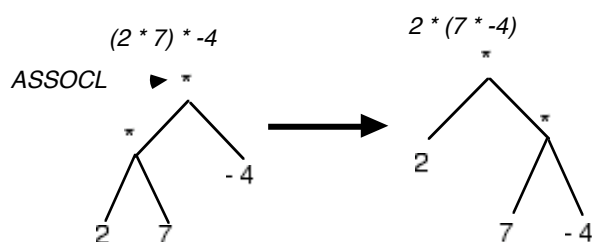
## Associativity

You can change the way that three numbers are grouped (or *associated*) when they are all added or when they are all multiplied without changing the expression's value.

Re-associate from left to right:  $(a + b) + c$  is the same number as  $a + (b + c)$   
 $(a * b) * c$  is the same number as  $a * (b * c)$

Re-associate from right to left:  $a + (b + c)$  is the same number as  $(a + b) + c$   
 $a * (b * c)$  is the same number as  $(a * b) * c$

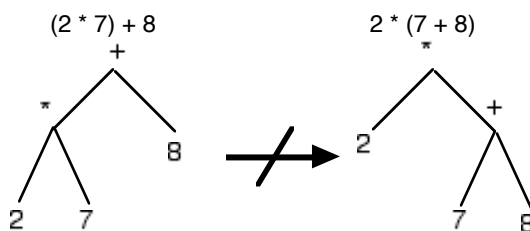
Example:



**ASSOCL** (“assoesh-L”) re-associates expressions from having the grouping on the left to having the grouping on the right, changing  $(2 * 7) * -4$  to the expression  $2 * (7 * -4)$ .

**ASSOCR** (“assoesh-R”) re-associates expressions from having the grouping on the right to having the grouping on the left, changing  $2 * (7 * -4)$  to the expression  $(2 * 7) * -4$ .

Not every set of operations on three numbers is associative. You cannot re-associate  $(2*7)+8$ . The expression  $(2*7)+8$  evaluates to 22. The expression  $2*(7+8)$  evaluates to 30. These expressions are not equivalent.



In order to apply associativity, the expression must have two operation signs and they must be both addition or both multiplication. That is, ASSOCL and ASSOCR can be applied only when you have a sum of a sum and something or a product of a product and something.

### Questions:

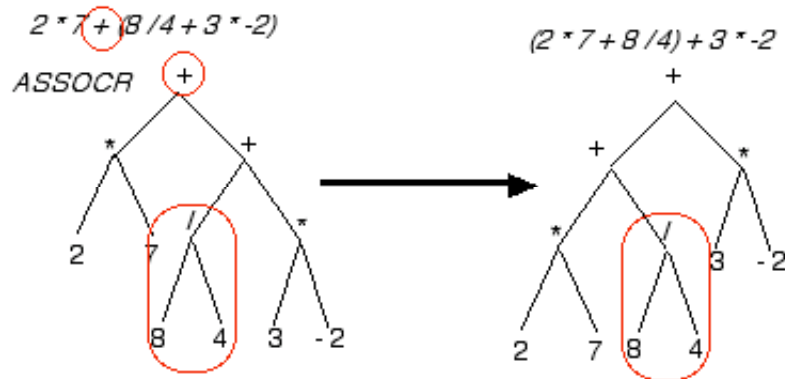
- Is division associative? That is, is  $(x / y) / z$  always equal to  $x / (y / z)$ ? Try some examples.
- What would it mean that subtraction is associative? Is subtraction associative?

- What would it mean that exponentiation is associative? Is exponentiation associative?

## Using Associativity of Addition and Multiplication with Complex Expressions

The operands in an expression do not have to be numbers in order to use associativity. The operands can be expressions, too.

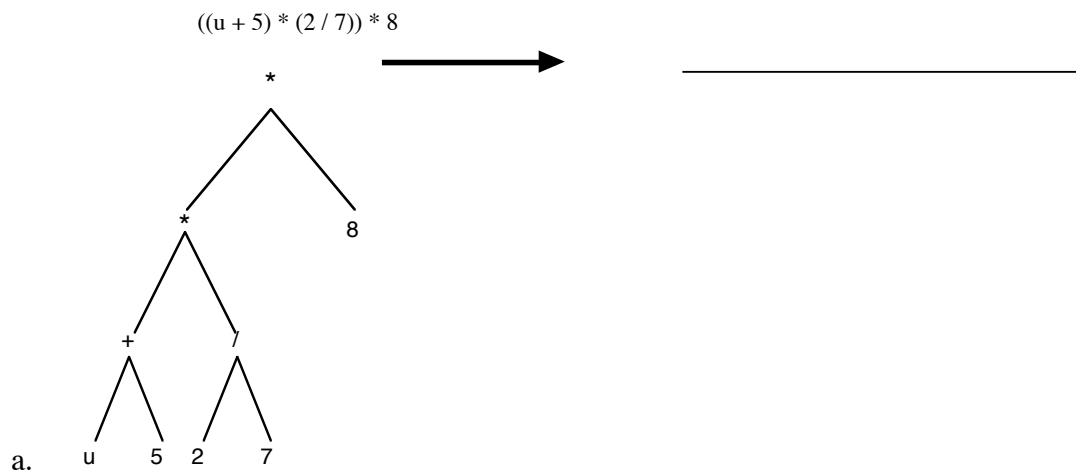
### Example

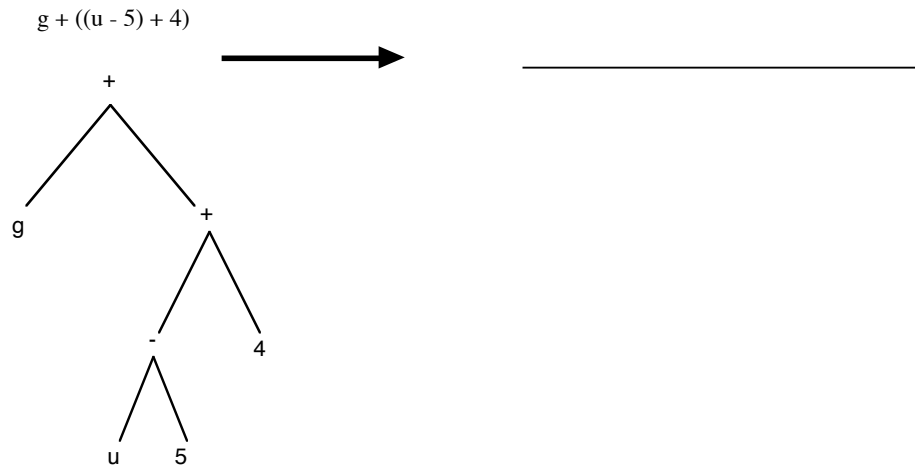


**ASSOCR** changed  $2 * 7 + (8 / 4 + 3 * -2)$  to  $(2 * 7 + 8 / 4) + 3 * -2$ . This follows the pattern of changing  $a + (b + c)$  to  $(a + b) + c$ . In this case  $2 * 7$  is  $a$ ,  $8 / 4$  is  $b$ , and  $3 * -2$  is  $c$ . All that matters is that you have three things, and that they are all added or all multiplied.

### Your Turn

Apply ASSOCL and/or ASSOCR to each of the following expressions. Circle the operation to which you will apply the ASSOCL or ASSOCR operation, then write the expression and draw the expression tree as they will appear afterward.





c. Circle the expressions to which neither ASSOCL nor ASSOCR can be applied. Explain why they cannot be used. It might be useful to draw expression trees before deciding.

i)  $a + b * 4$

ii)  $(x + y) + (u + v)$

iii)  $3 + k * (m * 5)$

iv)  $5 * y + u / t + 7$

## Exercises

In each of the exercises below, change the first expression into the second expression using COMMUTE, ASSOCL, and ASSOCR. On each line, circle the operation sign to which you would apply an action and write the action you would apply to it. Then write the resulting expression on the next line. Use the format shown in the example.

### Example

Start with:

Change it to:

$$1 + (2 + 3)$$

$$(3 + 1) + 2$$

Expression: \_\_\_\_\_  $1 + (2 \oplus 3)$   $\longrightarrow$  Action: COMMUTE

Expression: \_\_\_\_\_  $1 \oplus (3 + 2)$   $\longleftarrow$  Action: ASSOCL

Expression: \_\_\_\_\_  $(1 \oplus 3) + 2$   $\longleftarrow$  Action: COMMUTE

Expression: \_\_\_\_\_  $(3 + 1) + 2$   $\longleftarrow$  Action: \_\_\_\_\_

Start with:

Change it to:

$$((5 * 4) + 3) + 2$$

$$(3 + (4 * 5)) + 2$$

- 1) Expression:  $((5 * 4) + 3) + 2$  Action: \_\_\_\_\_
- 2) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 3) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 4) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 5) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 6) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 7) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 8) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 9) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 10) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 11) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_

Start with:

Change it to:

$$(2 + (-7 * 4)) + 3$$

$$(3 + 2) + (4 * -7)$$

- 1) Expression:  $(2 + (-7 * 4)) + 3$  Action: \_\_\_\_\_
- 2) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 3) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 4) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 5) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 6) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 7) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 8) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 9) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 10) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 11) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_



Start with:

Change it to:

$$(1 + 2) + (3 + 4)$$

$$(1 + 4) + (2 + 3)$$

- 1) Expression:  $(1 + 2) + (3 + 4)$  Action: \_\_\_\_\_
- 2) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 3) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 4) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 5) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 6) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 7) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 8) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 9) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 10) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 11) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_

Start with:

Change it to:

$$((8/3)*(2-8))*((7+6)*(3*2))$$

$$((8/3)*(3*2))*((2-8)*(7+6))$$

- 1) Expression:  $((8/3)*(2-8))*((7+6)*(3*2))$  Action: \_\_\_\_\_
- 2) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 3) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 4) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 5) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 6) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 7) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 8) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 9) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 10) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_
- 11) New Expression: \_\_\_\_\_ Action: \_\_\_\_\_

## Part II: Distributivity of Multiplication Over Addition

A statement that is always true about multiplication and addition is that

$$a * (b + c) \text{ and } (a*b) + (a*c)$$

are equivalent – they evaluate to the same number, regardless of the values of  $a$ ,  $b$ , and  $c$ .

This is easy to see when we think about rectangles. When we think of the rectangle below as being one rectangle, it has a height of  $a$  and a width of  $(b + c)$ . When we think of the same rectangle as broken into two smaller rectangles, its area is  $(a*b) + (a*c)$ . But the outer rectangles area is the same regardless of how we calculate it. So, we can say that  $a*(b+c)$  is the same number as  $(a*b) + (a*c)$ .



If we turn the statement around, we can say that

$$(a*b) + (a*c) \text{ and } a * (b + c)$$

are equivalent – they evaluate to the same number. Either way we state it, this property allows us to change expressions in these ways:

Starting with  $a*(b + c)$  and changing it to  $a*b + a*c$  is called distributing multiplication over addition from the left. We will call this operation DISTL.

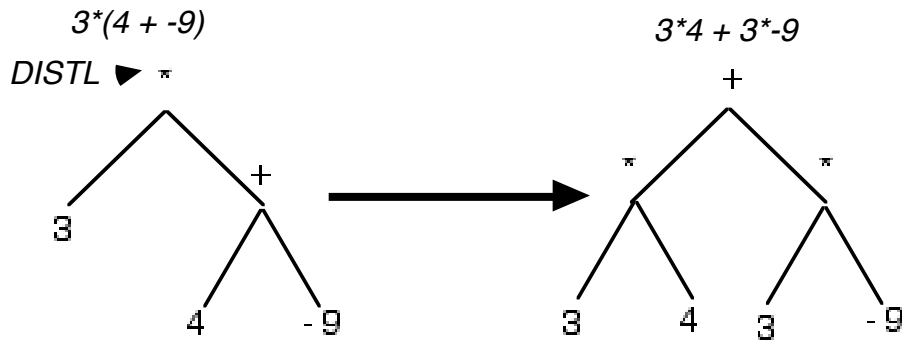
Starting with  $a*b + a*c$  and changing it to  $a*(b + c)$  is called collecting terms from the left. We will call this operation COLLECTL.

Distributing and collecting can also be done from the right.

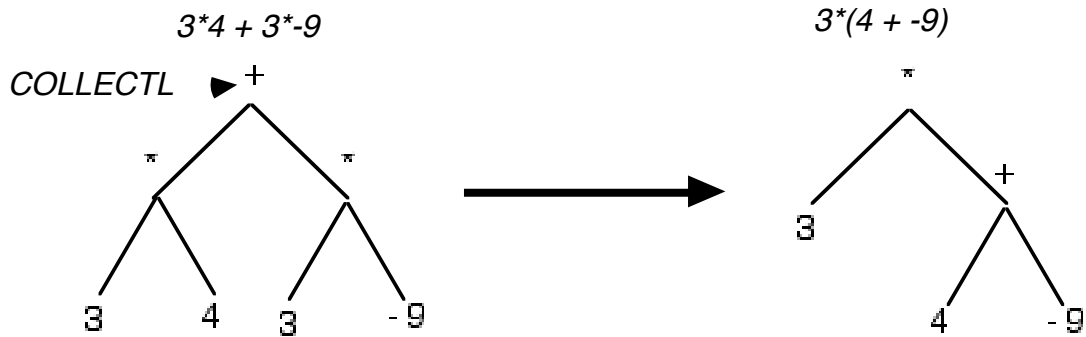
Starting with  $(b + c)*a$  and changing it to  $b*a + c*a$  is called distributing multiplication over addition from the right. We will call this operation DISTR.

Starting with  $b*a + c*a$  and changing it to  $(b + c)*a$  is called collecting terms from the right. We will call this operation COLLECTR.

DISTL (distributing multiplication over addition from the left) changes  $3*(4 + -9)$  into  $(3 * 4) + (3 * -9)$ . The tree diagrams below show this.



COLLECTL (collecting terms from the left) will undo the effect of DISTL, as shown below.

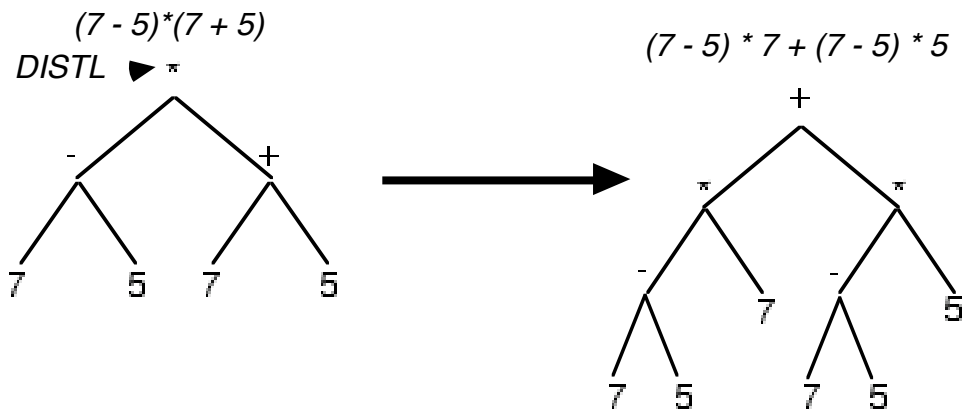


COLLECTL changes  $3*4 + 3*-9$  into  $3*(4 + -9)$ .

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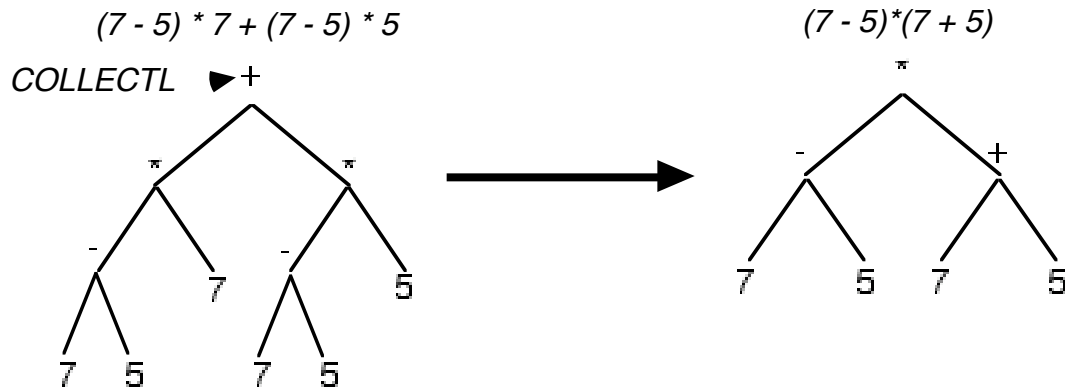
The letters  $a$ ,  $b$ , and  $c$  do not need to stand for numbers. They can stand for expressions, too.

*Example*



Here,  $(7-5)$  is distributed over  $(7 + 5)$ , giving  $(7 - 5)*7 + (7 - 5)*5$ .

COLLECTL will undo the effect of DISTL, as shown in the example below.



The expression  $(7-5)$  appears on the left in both products within the sum. Thus, COLLECTL will change the expression into  $(7 - 5) * (7 + 5)$ .

### Exercises

Rewrite each of the following expressions using DISTL, DISTR, COLLECTL, or COLLECTR. You may have to use COMMUTE or one of ASSOCL and ASSOCR before you can use distributivity.

1.  $x * 5 + s * 5$  \_\_\_\_\_
2.  $(9.5 + a) * (b + 7.5)$  \_\_\_\_\_
3.  $h * 4.3 + 4.3 * r$  \_\_\_\_\_
4.  $395 * (217 * 5) + (395 * 217) * 4$  \_\_\_\_\_
5.  $519 * (12457 * u + 9812 * u)$  \_\_\_\_\_

**Exercises**

In each of the exercises below, change the first expression into the second expression using COMMUTE, ASSOCL, ASSOCR, DISTL, DISTR, COLLECTL, or COLLECTR. On each line, circle the operation sign to which you would apply an action and write the action you would apply to it. Then write the resulting expression on the next line. Use the format shown in the example.

**Start with:****Change it to:**

$5 * (x + y)$

$y*5 + x*5$

Expression:  $5 * (x + y)$  Action: \_\_\_\_\_

New Expression: \_\_\_\_\_ Action: \_\_\_\_\_

New Expression: \_\_\_\_\_ Action: \_\_\_\_\_

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New Expression: \_\_\_\_\_ Action: \_\_\_\_\_

**Start with:**

**Change it to:**

$$5 * ((u + v) + 2)$$

$$(5*u) + ((2 + v)*5)$$

Expression: \_\_\_\_\_ Action: \_\_\_\_\_

Expression: \_\_\_\_\_ Action: \_\_\_\_\_

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Expression: \_\_\_\_\_ Action: \_\_\_\_\_

**Start with:**

**Change it to:**

$$(7 + 3)*(r + s)$$

$$(7*r + 7*s) + (3*r + 3*s)$$

Expression: \_\_\_\_\_ Action: \_\_\_\_\_

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Expression: \_\_\_\_\_ Action: \_\_\_\_\_

**Start with:**

**Change it to:**

$$(x + -5)*(x + 5)$$

$$(x*x) + (5*x) + (-5*x) + (-5*5)$$

Expression: \_\_\_\_\_ Action: \_\_\_\_\_

Expression: \_\_\_\_\_ Action: \_\_\_\_\_

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**Start with:**

**Change it to:**

$$3*(a + b) + 9*(b + a)$$

$$(9 + 3)*(b + a)$$

Expression: \_\_\_\_\_ Action: \_\_\_\_\_

Expression: \_\_\_\_\_ Action: \_\_\_\_\_

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